Overview: Studying linear maps.

$$\begin{tikzcd}
V_B \arrow[r, "\mathrm{Rep}_{B,D}(L)"] & W_D \arrow[d, "\mathrm{Rep}_{D,D'}(id)"] \\
V_{B'} \arrow[u, "\mathrm{Rep}_{B',B}(id)"] \arrow[r, "\mathrm{Rep}_{B',D'}(L)"'] & W_{D'}
\end{tikzcd}$$

$$\mathrm{Rep}_{B',D'}(L) = \mathrm{Rep}_{D,D'}\,id \cdot \mathrm{Rep}_{B,D}(L) \cdot \mathrm{Rep}_{B',B}(id)$$

NB: The order of multiplication of matrices DOES Matter...

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$(f \circ g)(x) = f(g(x))$

$A \xrightarrow{g} B \xrightarrow{f} C$

Defn: A matrix A is similar to matrix B when there is an invertible matrix P with $B = P^{-1}AP$.

Ex: $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

So $P^{-1} = \frac{1}{1\cdot 1 - 0\cdot 1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ inverse formula for 2×2 matrices.

Then $B = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ is similar to A.

(by definition)

NB: Similarity of $n\times n$ matrices is an equivalence relation:

① Every matrix is similar to itself. $(A = I^{-1} A I)$ ✓

② If A is similar to B, then B is similar to A. ✓

$\left(\text{If } B = P^{-1}AP, \text{ then } PB = AP, \text{ so } PBP^{-1} = A\right)$

③ If A is similar to B and B is similar to C, then A is similar to C. ✓

$\left(\text{if } B = P^{-1}AP \text{ and } C = Q^{-1}BQ, \text{ then}\right.$

$\left. C = Q^{-1}BQ = Q^{-1}(P^{-1}AP)Q = (PQ)^{-1}A(PQ)\right).$

Q: When are two matrices similar?

A: A and B are similar when they represent the same linear operator w.r.t. different bases.

$$
\begin{tikzcd}
\mathbb{R}^n_B \arrow[r, "A"] \arrow[d, "P^{-1}"] & \mathbb{R}^n_B \arrow[d, "P^{-1}"] \\
\mathbb{R}^n_D \arrow[r, "C"'] \arrow[u, "P"] & \mathbb{R}^n_D
\end{tikzcd}
$$

$P = \mathrm{Rep}_{D,B}(id)$

$C = P^{-1}AP$

Point: Similarity is all about basis change!

Ex: Let $L_0 : \mathbb{R}^3 \to \mathbb{R}^3$ take $L_0\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+y+z\\y+z\\z\end{pmatrix}$

and $L_1 : \mathbb{R}^3 \to \mathbb{R}^3$ take $L_1\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2x+y-z\\x-y+z\\z\end{pmatrix}$.

w.r.t. $\mathcal{E}_3$ we have $\mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(L_0) = \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} = M$.

OTOH $\mathrm{Rep}_{\mathcal{E}_3,\mathcal{E}_3}(L_1) = \begin{bmatrix}2&1&-1\\1&-1&1\\0&0&1\end{bmatrix} = N$. Now we compute the determinants of M and N:

$$\det(M) = \det\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 1.$$

$$\det(N) = \det\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = 0 - 0 + 1\det\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= 2\cdot -1 - 1\cdot 1 = -3$$

So M and N are <u>not</u> similar. $\square$

<u>NB</u>: If M is similar to N, then $M = P^{-1}NP$ implies

$$\det(M) = \det(P^{-1}NP) = \det(P^{-1})\det(N)\det(P)$$

$$= \frac{1}{\cancel{\det(P)}}\det(N)\cancel{\det(P)} = \det(N).$$

<u>Ex</u>: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ both have $\det(I_2) = 1$ and $\det(J) = 1$, but I_2 and J are <u>not</u> similar... For every P, invertible:

$$P^{-1}I_2P = P^{-1}P = I_2,$$ so I_2 is <u>NOT</u> similar to J.

Q: When is a matrix M similar to a diagonal matrix?

EIGENVECTORS AND EIGENVALUES

<u>Def^n</u>: A linear operator L has <u>eigenvector</u> $0 \neq v \in \mathrm{dom}(L)$ with <u>eigenvalue</u> λ when $L(v) = \lambda v$.

<u>Prop</u>: Given eigenvalue λ for L, the eigenspace $V_\lambda = \{v \in \mathrm{dom}(L) : L(v) = \lambda v\}$ is a subspace of $\mathrm{dom}(L)$.

Method to compute eigenvalues of M ($\leftarrow$ rep. a lin map).

① Compute characteristic polynomial $p_M(\lambda) = \det(M - \lambda I)$.

② Compute roots of $p_M(\lambda)$. (i.e. solve $p_M(\lambda) = 0$).

③ Those roots are all the eigenvalues!

$$\left(\begin{array}{ccc} (M - \lambda I)v = \vec{0} & \Longleftrightarrow & \boxed{Mv = \lambda v} \\ \Updownarrow & & \uparrow \\ \underline{\det(M - \lambda I) = 0} & & \text{some } v \neq \vec{0} \end{array} \right) \star$$

Ex: Let $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. $p_M(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = \boxed{(1-\lambda)^2}$

has roots $\lambda = 1$. Thus M has eigenvalue $\lambda = 1$. $\boxtimes$

Defn: The *algebraic multiplicity* of eigenvalue $\lambda = \alpha$ is the corresponding power of $\lambda - \alpha$ in a complete factorization of $p_M(\lambda)$.

Recall: Polynomial $f(x)$ has $f(\alpha) = 0$ iff $x - \alpha$ is a factor of $f(x)$...

NB: In the example above $\left(M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$, $\lambda = 1$ had algebraic multiplicity 2.

Q: How do we compute eigenspaces (i.e. the eigenvectors)?

Method to Compute Eigenspaces:

① Compute eigenvalues via $p_M(\lambda) = 0$.

② The eigenspace associated to $\lambda = \alpha$ is precisely $\mathrm{null}(M - \alpha I)$ (i.e. $V_\alpha = \mathrm{null}(M - \alpha I)$).

Ex: For $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $P_M(\lambda) = (1-\lambda)^2$.

$\lambda = 1$: $\text{null}\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = \text{null}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = V_1$

$\text{RREF}(M-\lambda I) = \text{RREF}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

yields null space: $\begin{bmatrix} x \\ y \end{bmatrix} \in \text{null}(M - \lambda I)$ iff $y = 0$

i.e. $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is a basis of V_1. $\boxtimes$

Ex: Let $M = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

$$P_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & \boxed{3-\lambda} & 0 \\ 2 & 0 & 1-\lambda \end{bmatrix}$$

$$= -0 + (3-\lambda)\det\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} - 0$$

$$= (3-\lambda)\left((1-\lambda)^2 - 4\right)$$

$$= (3-\lambda)\left((1-\lambda)^2 - 2^2\right)$$

$$= (3-\lambda)(1-\lambda-2)(1-\lambda+2)$$

$$= (3-\lambda)(-1-\lambda)(3-\lambda) = (3-\lambda)^2(-1-\lambda).$$

$\therefore$ eigenvalues $\lambda = 3$ and $\lambda = -1$ w/ algebraic mult. 2 and 1 respectively. Now the eigenspaces:

$\lambda = 3$: $M - 3I = \begin{bmatrix} 1-3 & 0 & 2 \\ 0 & 3-3 & 0 \\ 2 & 0 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix}$

which has $\text{RREF}(M - 3I) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$\therefore$ $\text{null}(M-3I) \ni \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ iff $x - z = 0$ iff $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$

thus $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is a basis of $\text{null}(M-3I) = V_3$.

$\underline{\lambda=-1:}$ $M+I = \begin{bmatrix} 1-(-1) & 0 & 2 \\ 0 & 3-(-1) & 0 \\ 2 & 0 & 1-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}$ which has

$\text{RREF}(M+I) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so we have computed

$\begin{bmatrix}x\\y\\z\end{bmatrix} \in V_{-1}$ iff $\begin{cases} x + z = 0 \\ y = 0 \end{cases}$ iff $\begin{cases} x = -t \\ y = 0 \\ z = t \end{cases}$ iff $\begin{bmatrix}x\\y\\z\end{bmatrix} = t\begin{bmatrix}-1\\0\\1\end{bmatrix}$.

So V_{-1} has basis $\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$. ☒

$\underline{\text{Defn}}$: The $\underline{\text{geometric multiplicity}}$ of eigenvalue $\lambda = \alpha$ is the dimension of the eigenspace V_α.

(i.e. geom mult $= \dim(V_\alpha)$).

$\underline{\text{NB}}$: In the example above, 3 has 2 = geom mult = alg mult and -1 had 1 = geom mult = alg mult.

$\underline{\text{Ex}}$: $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ had $p_M(\lambda) = (1-\lambda)^2$ but $\dim(V_1) = 1 \neq 2$.

So geometric mult does $\underline{\underline{\text{NOT}}}$ always agree w/ alg mult. ☒